For all of the problems below, let \( N \) be the set of naturals, \( N = \{0, 1, 2, 3, 4...\} \)

**Problem 1: Countability of rationals**

Consider the set \( Q \) of rational numbers. Recall the definition: a number \( x \) is rational if there exists natural \( a,b \) such that \( x = a / b \).

**a)** Find an ONTO mapping \( m \) from \( N \times N \) to \( Q \). Prove that \( m \) is ONTO, using the definition of rational numbers.

**b)** Use \( m \) together with results we saw in class to prove that \( Q \) is countable. Make sure to explicitly state which results that we showed in class you are using in your proof!

**Problem 2: Countability of triples**

Let \( T \) be the set of all triples of naturals, \( T = N \times N \times N \). (We saw it before on the last homework).

Show that \( T \) is countable as follows:

**a)** find a way to list all its members so each member of \( T \) has a finite index in your list.

**HINT:** in the last homework, we partitioned \( T \) into subsets \( T_0, T_1, T_2, T_3, \ldots \) such that \( T_k \) consists of triples from \( T \) whose members sum up to \( k \):

\[
T_k = \{ (a, b, c) \in T \mid a+b+c = k \}
\]

For example, \( T_2 = \{ (0,1,1), (1,0,1), (1,1,0), (0,0,2), (0,2,0), (2,0,0) \} \)

**b)** show that your listing gives a bijection between \( T \) and \( N \), thereby proving that \( T \) is countable

**Problem 3. Strings of bits and letters**

In class we’ve worked with the set IBS, infinite bit strings. Now consider the set ILS, infinite letter strings. These are infinite strings of letters \( \{a..z\} \) such as “aaaaaaaaa...”, “csecsecse...”, etc.

**a)** find a 1-1 mapping \( m_I \) from IBS to ILS, show what this mapping will do to the following infinite bit strings

\[
m_I ("10101010101...")
m_I ("11110000000...")
m_I ("00000000000...")
\]

**HINT:** can we use letters to stand in for bits? Remember, the mapping does NOT need to be onto!
b) find a 1-1 mapping \( m_2 \) from ILS to IBS, show that what this mapping will do to the following infinite letter strings

\[
\begin{align*}
& m_2 (\text{"aaaaaaaaaa"}) \\
& m_2 (\text{"cesecese"})
\end{align*}
\]

HINT: can we represent the letters in binary code?

c) In class, we saw that IBS is uncountable; it has the same cardinality as REALS. Based on (a) and (b), what can you conclude about the cardinality of ILS? Prove your answer.

**Problem 4. Multiplying Alephs**

Consider the following bijection \( m \) between REALS and INTS \( \times \) 01REALS: given any real \( x \), \( m(x) = (a, b) \) where:

- \( a \in \text{INTS} \) is the part of \( x \) before the decimal point
- \( b \in \text{01REALS} \) is the part after the decimal point

Example: \( m(\pi) = (3, 0.14159...) \); \( m(-17.0) = (-17, 0.00000...) \)

Use \( m \) together with results we saw in class to prove the following equality:

\[
\aleph_0 * \aleph_1 = \aleph_1
\]

**Problem 5. Proof by diagonalization**

a) In class we saw a proof by diagonalization that there is no bijection between \( \mathbb{N} \) and 01REALS, and concluded that 01REALS are not countable.

Use the same argument to prove by diagonalization that there is no bijection between \( \mathbb{N} \) and ILS, the set of infinite letter strings.

b) In class we saw a proof by diagonalization that there is no bijection between \( \mathbb{N} \) and \( \mathcal{P}(\mathbb{N}) \), and concluded that \( \mathcal{P}(\mathbb{N}) \) is not countable.

Use exactly the same argument to prove by diagonalization that there is no bijection between \( \mathbb{N} \) and \( \mathcal{P}'(\mathbb{N}) \), where \( \mathcal{P}'(\mathbb{N}) \) is a set consisting of all finite sets of naturals:

\[
\mathcal{P}'(\mathbb{N}) = \{ x : x \text{ is a finite subset of } \mathbb{N} \}
\]

Pay close attention to your argument – at some point, it should crash into a logical wall and fail, because \( \mathcal{P}'(\mathbb{N}) \) is actually countable.

*Explain carefully where this proof fails and why.*