Independent events

DEFINITION: events A, B with P(A), P(B) are independent iff P(A \cap B) = P(A) * P(B)

Example: a family with 5 children
A = there are 3 boys out of 5, B = the oldest of 5 is a boy (male)
P(A) = \frac{C(5,3)}{32} = \frac{5}{16}
P(B) = \frac{1}{2}

Consider A \cap B - there are 3 boys (males) out of 5 AND the oldest is a boy (male)
what's the probability that A AND B will happen? P(A \cap B)
Note: if A and B are independent, then P(A \cap B) = P(A) * P(B)

E1 = A \cap B = \{(M_ _ _ _ )\} -- 2 more M's among the 4 blanks; C(4,2) = 6 = N(E)
P(E1) = P(A \cap B) = \frac{6}{32} = \frac{3}{16}

are A and B independent?? P(A) * P(B) = \frac{5}{16} * \frac{1}{2} = \frac{5}{32} \neq P(A \cap B) = \frac{6}{32} \ NO!!!
Having 3 boys out of 5 and having the oldest boy are not independent!!

now consider the event E2 = A \cap !B – there are 3 boys out of 5 and the oldest of 5 is a female
E2 = A \cap !B = \{(F_ _ _ _ )\} -- 3 more M's among the 4 blanks; C(4,3) = 4 = N(E2)
P(E2) = P(A \cap !B) = \frac{4}{32} = \frac{1}{8}
This shows that P(E1) \neq P(E2)

NOTE: if A and B were independent events, P(E1) would be equal to P(E2)
This is because P(B) = P(!B)

intuition: does having a specific value for event A affect probability for event B?
if yes, then not independent!

Working with tables

SOE students

<table>
<thead>
<tr>
<th>TALL MALES -- 400</th>
<th>TALL FEMALES -- 150 TALL</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHORT MALES -- 200</td>
<td>SHORT FEMALES -- 250 SHORT</td>
</tr>
</tbody>
</table>

TOTAL MEN - 600, TOTAL WOMEN - 400
% of a random person being male = 60%, female = 40%

TOTAL TALL = 550, total short - 450
% of a random person being tall = 55%, short = 45%
what is the probability of a random SOE student being a tall male???
(if independent) \( .6 \times .55 = .33 \)
but it's not -- it's 0.4
not independent BECAUSE % of men who are tall is different from % of women who are tall

**Probability trees**

```
60% male  \[ \overset{.4}{\text{tall}} \]
\quad \leftarrow \quad 0.4 \quad \overset{.2}{\text{medium}} \\
\quad \downarrow \quad \quad \quad \downarrow \\
\quad \text{person} \quad \text{.5 medium} \\
\quad \quad \quad \downarrow \\
40% female \quad \overset{.2}{\text{tall}} \\
\quad \quad \quad \downarrow \\
\text{40% female} \quad \overset{.3}{\text{short}}
```

best for sequential events, but not necessary
at any node, P of all children adds up to 1
any number of children is possible!

what is the probability of randomly picking a tall male?
Take the leaf of interest (circled above), multiply the probabilities for each one
\[ .6 \times .4 = .24 \]

This also works when the answer involves multiple leaves
what is the probability of randomly picking a tall person?
There are two leaves of interest – tall male and tall female
for male: \( .6 \times .4 = .24 \)
for female: \( .4 \times .2 = .08 \)
answer: \( .24 + .08 = .32 \)

Another example: You have 4 best friends A B C D
your chance of running into a is 10%, b 20%, c 20%, d 50%
a wears red shirt EVERY day, b every other, c every third, d every fourth
what's the chance the first friend you running into will wear a red shirt?

create probability tree!

```
10% a  \[ \overset{100\% \text{ red}}{\text{100% red}} \]
\quad \leftarrow \quad 0 \% \text{ not} \\
\quad \downarrow \quad \quad \quad \downarrow \\
\quad 20\% b \quad \overset{50\% \text{ red}}{\text{50% red}} \\
\quad \quad \quad \downarrow \\
\quad \text{friend} \quad \overset{50\% \text{ not red}}{\text{50% not red}} \\
\quad \quad \quad \downarrow \\
\quad 20\% c \quad \overset{1/3 \text{ red}}{\text{1/3 red}} \\
\quad \quad \quad \downarrow \\
\quad \text{friend} \quad \overset{2/3 \text{ not red}}{\text{2/3 not red}} \\
\quad \quad \quad \downarrow \\
\quad 50\% d \quad \overset{1/4 \text{ red}}{\text{1/4 red}} \\
\quad \quad \quad \downarrow \\
\quad \text{friend} \quad \overset{3/4 \text{ not red}}{\text{3/4 not red}}
```
what's the chance the first friend you running into will wear a red shirt?
add up the probabilities for all leaves of interst!!
\[ .1 \times 1 + .2 \times .5 + .2 \times \frac{1}{3} \times .5 \times \frac{1}{4} = 1/10 + 1/10 + 1/15 + 1/8 = \text{about .39} \]
to be exact, find common denominator which is 120
\[ \frac{12}{120} + \frac{12}{120} + \frac{8}{120} + \frac{15}{120} = \frac{47}{120} \]

another example:
if you go outside, what's the chance that the first random person you see wears a red shirt?
Let's say it's 10%

based on this, what's the chance that the first two random people you see wearing a red shirt?
You would think it's \(.1 \times .1 = 1\%\)

What about first 10 people? \((.1)^{10} = .0000000001\) – very very very low
but tomorrow you step outside and you see a marching band with 10 people in red uniforms!!!

what happened here?
if people always made the decision what to wear totally individually, then it would be
independent and \((.1)^{10} = .0000000001\) would be correct

But if they are both members of the same team/band/employee/army - SAME UNIFORM
so these events are not independent!!

Last example:
3 types of people A, B, C (i.e. age groups, or blood types, or intelligence levels, etc)
medical condition M - 30% of population
if types and condition were independent, then 30% of As would have it, 30% of Bs and 30% of Cs
but may be dependent; for example, no A's have it, 1/3rd B's, 2/3 C's

can use table: type of person vs. having condition
or can use probability trees
what;s % of people are A's with condition M? 0%
what;s % of people are B's with condition M? \(1/3 \times x\) where \(x\) is the % of B's in pop

**Conditional probability**

We have dependent events A and B

Conditional probability \(P(B \mid A) = \text{"probability of B given A"}\)

We've seen lots of examples already, from probability trees:

- probability of condition M given age group C
- probability of red shirt given friend B
- probability of tall given male

NOTE: if A and B are independent, \(P(B \mid A) = P(B)\), since A has no effect on B

We are now ready for the proper definition of \(P(A \cap B)\):
\[ P(A \cap B) = P(A) \times P(B \mid A) \]
Intuition: think of probability trees – this is exactly how we computed $P(A \cap B)$ there!

NOTE: if $A$ and $B$ are independent, $P(B \mid A) = P(B)$, so

$$P(A \cap B) = P(A) \times P(B \mid A) = P(A) \times P(B)$$

So this definition reduces to the old one in the special case when $A$ and $B$ are independent!

Recall that $A \cap B = B \cap A$
So $P(B \cap A) = P(A \cap B) = P(B) \times P(A \mid B) = P(A) \times P(B \mid A)$

Divide both sides by $P(B)$ and you get this:

$$P(A \mid B) = \frac{P(B \mid A) \times P(A)}{P(B)}$$

THIS FORMULA IS REALLY USEFUL FOR DOING CERTAIN TYPES OF REASONING IN AI

given your symptoms, we are looking for most likely disease
given how the car is misbehaving, what is the most likely problem with it

EXPERT SYSTEM: given some observation $A$ in your "world", find the cause $B$
this is the same as finding specific value for $B$ such that $P(B \mid A)$ is the highest
the "world" can be a patient, a car, or a computer system

PUZZLE: monty hall problem

Suppose you're on a game show, and you're given the choice of three doors
Behind one door is a car; behind the others, NOTHING

you pick random door, 1/3rd chance of car
(but no matter what you pick, one of the other doors is guaranteed to be empty)
then host opens another door, and it's EMPTY
(note: that door was not chosen at random! if one had the car, he picked ANOTHER)
should you switch to the other door – the one he did not open??
The answer is YES

TAKEAWAY: extra information changes probabilities