READING:

Section 9.2
Section 9.3

PROBABILITY

what's the probability that X will happen? P(X)
example - there are 3 boys out of 5, P(X) = C(5,3)

what's the probability that Y will happen? P(Y)
example - the oldest is a boy, P(Y) = 1/2

what's the probability that both X and Y will happen? P(X \cap Y)
there are 3 boys out of 5 AND the oldest is a boy

what's the probability that either X OR Y will happen? P(X \cup Y)
there are 3 boys out of 5 OR the oldest is a boy

X, Y - events
all possible outcomes - sample space

P(X) = N(X) / N(S)

Probability for union of events

P (A \cup B) = P(A or B) = P(A) + P(B) - P(A \cap B) = always true

For intuition, think of probabilities vs. areas

if lots of outcomes evenly distributed through the sample space
then P(E) = A(E) / A(S) ratio of areas

Area (A \cup B) = Area(A) + Area(B) - Area(A \cap B)

P (A \cup B) = P(A) + P(B) - P(A \cap B) = always true

Probability for intersection of events

P (A \cap B) = P(A and B) = P(A) * P(B)
Note: multiplication rule only works when events are *independent*
I.E. the value of one is not connected to the value of the other
I.E. knowing the value of one does not change the likelihood of the value of the other
we'll learn the proper definition soon

Multiplication rule applies to any number of *independent* events E1, E2, E3, ...

\[ P(E_1 \cap E_2 \cap E_3 \ldots) = P(E_1) * P(E_2) * P(E_3) \ldots \]

Note: they may be at the same time (parallel) or in sequence

Example: tossing 1 dice 4 times (sequence) vs. tossing 4 dice at once (parallel)
P(all four 6's) = 1/6^4

**There are two ways to get this answer**

What we learned last week:

think of all strings of length 4 of digits 1..6
\[
N(S) = \text{total number of strings} = 6^4 \quad N(E) = 1, \text{answer is } 1/6^4
\]

event space is "6666", so \(N(E) = 1\)
answer = \(N(E) / N(S) = 1/6^4\)

New way: multiplying probabilities, where \(P(\text{single toss} = 6) = 1/6\)

1\textsuperscript{st} toss is 6 = 1/6
2\textsuperscript{nd} toss is 6 = 1/6
3\textsuperscript{rd} toss is 6 = 1/6
4\textsuperscript{th} toss is 6 = 1/6
multiplying these together = 1/6^4

Some examples of independent events:

E1 – SOE student is female, E2 – SOE student is a genius
E1 – coin toss 1 is a 6, E2 - coin toss 2 is a 6

Some examples of dependent events:

E1 – SOE student is female, E2 – SOE student is tall
E1 – it is raining, E2 – I am carrying an umbrella

**Empty event intersection**

Recall the rule for union of events: \(P(A \cup B) = P(A) + P(B) - P(A \cap B)\)

When \(P(A \cap B) = 0\), this rule becomes simpler: \(P(A \cup B) = P(A) + P(B) !!\)

\(P(A \cap B) = 0\) when \(N(A \cap B) = 0\), meaning \(A \cap B\) is empty
this means that \(A\) and \(B\) never occur together, they are *disjoint*
Scenario of outcomes that are not equally likely

when the various outcomes in S are NOT equally likely, and E ⊆ S, computing P(E) changes
we cannot just take the size of E and S and return the ratio
think of all the elements as having weights
think of N(E) and N(S) as measuring total weight rather than total count
if everyone has weight 1, computing count is the same as computing weight
but otherwise it's not....

example:
bias coin, twice as likely to be heads
consider event H “we have heads”
when it's not heads it's tails, so T = !H = “we have tails”

\[
P(H) + P(T) = P(H) + P(!H) = 1 \quad \text{– this is always the case for any event}
\]

Since the coin is twice as likely to be heads, \( P(H) = 2/3; P(T) = 1/3 \)

How about two coin tosses? \( S = \{HH, TT, HT, TH\} \)

\[
\begin{align*}
P(HH) &= (\text{use multiplication rule!}) = 2/3 \cdot 2/3 = 4/9 \\
P(TH) &= 1/3 \cdot 2/3 = 2/9 \\
P(TT) &= 1/3 \cdot 1/3 = 1/9 \\
P(HT) &= P(TH) = 2/9
\end{align*}
\]

\( P(HH \text{ or } TH \text{ or } TT \text{ or } HT) \text{ should be 1 } \)
use the sum rule!! =
\( P(HH \text{ or } TH \text{ or } TT \text{ or } HT) = P(HH) + P(TH) + P(TT) + P(HT) = 4/9 + 2/9 + 1/9 + 2/9 = 9/9 = 1 \)
Note: this is because these four possibilities are disjoint (can't have both heads and tails at once)

\[
\begin{align*}
P(\text{there is at least one } H) &= P(HH \cup TH \cup HT) = 8/9 \\
P(\text{there is at least one } T) &= P(TT \cup TH \cup HT) = 5/9 \\
\end{align*}
\]
They are not the same because of the coin bias

WHAT IF I TOSSED THIS COIN 12 TIMES?
\( P(\text{all tails}) = (1/3)^{12} \approx 1.88 \cdot 10^{-6} \)
\( P(\text{there is at least one } H) = 1 - P(\text{all tails}) = 1 - (1/3)^{12} \approx .99 \)

Similarly, \( P(\text{there is at least one } T) = 1 - P(\text{all heads}) = 1 - (2/3)^{12} = .992 \)
TAKEAWAY: sometimes, it's easier/better to compute \( P(\text{complement event})! \)