REVIEW:
COMBINATORICS answers questions of the type:
How many combinations are there ...
How many ways are there to choose ....

SCENARIO 1. order, repetitions
ex: a string of length n over k chars
answer: k^n

SCENARIO 2. order, no repetitions
ex: a list of length n from k unique values, k ≥ n
answer k! / (k-n)!

SCENARIO 3. no order, no repetitions
ex: a team of n players from k people, k ≥ n
answer k! / (k-n)! n!

k CHOOSE n = k! / (k-n)! n!
Written as follows:

(k
\binom{n}{k}

pretend k and n are under each other

Note: in the textbook, they talk about "n choose r"

ex: 8 choose 5 = 8! / 3! 5! =

8 * 7 * 6 * 5 * 4 * 3 * 2 * 1 / 3 * 2 * 1 * 5 * 4 * 3 * 2 * 1 = 56

Note: always start by cancelling out as much as you can!

Note: 8 choose 5 = 8! / 3! 5! = 8! / 5! 3! = 8 choose 3

This can be generalized: n choose r = n! / (n-r)! r! = n! / r!(n-r)! = n choose (n-r)
There is a SYMMETRY

There is a simple logic for this symmetry; set of n values, have to take r of them, leave n-r
n choose 0 = 1, n choose n = 1
n choose 1 = n = n choose (n-1)
n choose 2 = n! / (n-2)! / 2! = n (n-1) / 2

**binomial distribution:**
graphing n choose 0 … n, as n → infinity
aka BELL CURVE

These are also coefficient values for the expansion of \((x + 1)^n\)

\[(x + 1)^n = 1 \cdot x^n + n \cdot x^{n-1} + \binom{n}{2} \cdot x^{n-2} + \ldots + n \cdot x + 1 \cdot x^0\]

the coefficients are: n choose 0, n choose 1, n choose 2; … n choose n-2, n choose n-1, n choose n

Examples:

\[(n+1)^2 = x^2 + 2n + 1\]

\[(n+1)^3 = 1 \ 3 \ 3 \ 1\] (these are just the coefficients)

\[(n+1)^4 = 1 \ 4 \ 6 \ 4 \ 1\] (these are just the coefficients)

\[4 \text{ choose } 2 = 4! / 2! 2! = 4 \cdot 3 \cdot 2 \cdot 1 / 2 \cdot 2 = 6\]

**NEW SCENARIO**

How many ways are there to create a bit string of length n with k 1's?

= # ways are there to create a bit string of length n with k 0's
= # ways are there to create a bit string of length n with (n-k) 1's
= # ways are there to create a bit string of length n with (n-k) 0's
(if there are n-k 1's then there are k 0's)

Example: string of length 5 with 3 1's and 2 0's

first 1 - choose a location for it (index in the string) - 5 choices
second 1 - 4 choices
third 1 - 3 choices
0's go in the remaining spots (no choice)
But now divide by the number of permutations of 1's because they are all the same!!

answer: \(5 \cdot 4 \cdot 3 / 3! = 60 / 6 = 10 = 5 \text{ choose } 3\)
Note: We could have also chosen spots for 0's instead of 1's
  first 0 - choose a location for it (index in the string) - 5 choices
  second 0 - 4 choices
  1's go in the remaining spots (no choice)
But now divide by the number of permutations of 0's because they are all the same!!!

answer: $5 \times 4 / 2! = 20 / 2 = 10 = 5$ choose $2 = 5$ choose $3$
In general: string of length $n$ with $r$ 1's and $(n-r)$ 0's
  first 1 - choose location - $n$ choices
  second 1 - $n-1$ choices
  third 1 - $n-2$
  ...
  $r$'th one - $n-r+1$ choices
multiply them out, you get $n! / (n-r)!$
But the divide by number of permutations of $r$ which is $r!$
The final answer is $n! / (n-r)! \times r!$ which is $n$ choose $r$

ANOTHER SCENARIO

how many ways to choose a team of 5 people out of 8 (5 men and 3 women)
if we need 3 men and 2 women on the team?

Think of it as choosing 2 subteams:
  3 men out of 5 men --- 5 choose 3 choices --- $5 \times 4 \times 3 \times 2 \times 1 / 3 \times 2 \times 1 = 10$
  2 women out of 3 women -- 3 choose 2 choices -- 3
what's the final answer?? 3 * 10 = 30

MULTIPLICATION

We multiply the number of combinations if there several independent choices that all have to be
made, and we are looking for all combinations of all choices

The answer is product of # of combinations for each choice

ANOTHER SCENARIO

How many ways to choose a team of 5 or 6 people out of 8?
# teams of five = 8 choose 5 = $X$
# teams of six = 8 choose 6 = $Y$
The answer is $X+Y$

Set of possible teams = UNION of the set of 5-teams and set of 6-teams

How many ways are there to get a team of 5 with at least 3 boys, if there are 7 boys and 10 girls?
Set of possible teams = UNION of all teams with 3 boys, all teams with 4 boys, and all teams with 5 boys
These sets of teams are disjoint!
So the # of possible teams = sum of the size of each set
3 boys, 2 girls - \( X = (7 \text{ choose } 3) \times (10 \text{ choose } 2) \)
4 boys, 1 girl - \( Y = (7 \text{ choose } 4) \times (10 \text{ choose } 1) \)
5 boys, 0 girls - \( Z = (7 \text{ choose } 5) \times (10 \text{ choose } 0) \)
answer - \( X + Y + Z \)

LAST SCENARIO

How many ways to rearrange the letters of MISSISSIPPI??
11 LETTERS: 4 S's, 4 I's, 2 P's, 1 M

Note: the answer is not 11! it's smaller

There are 2 approaches to get answer!

Approach 1: by positions - recall bit string positions
- \((11 \text{ choose } 4)\) to place S = x
- \((11-4 = 7 \text{ choose } 4)\) to place I = y
- \((7 – 4 = 3 \text{ choose } 2)\) to place O = z
- \((3-2 = 1 \text{ choose } 1)\) to place M = q
answer x * y * z * q
We have multiplication here, not addition!

Approach 2: think of a mapping

Domain: all permutations of mississippi
Codomain: all lists without unique letters
as if the letters are unique

\{S-3 S-4 I-5 S-6 S-7 I-8 P-9 P-10 I-11 M-1 I-2 \} ----> S I S S I P P I M I

SIZE OF DOMAIN IS 11!
WE ARE INTERESTED IN THE SIZE OF CODOMAIN

There is a COLLAPSE where words with the same positions for S are mapped to the word 4! words collapse to one word

There is a COLLAPSE where words with the same positions for I are mapped to the word 4! words collapse to one word

There is a COLLAPSE where words with the same positions for P are mapped to the word 2! words collapse to one word

ANSWER = \( \frac{11!}{4! \times 4! \times 2!} \)
Note: if you do out the math, you will see that the two approaches give the same answer