Reading (in textbook, unless stated otherwise):

- Sequences: chapter 5 through p. 263
- Sum of 1..k: middle of p. 277 through ex. 5.2.2

NOTE: where the book differs from lecture material, please follow the lecture material instead; for example the textbook puts the end index on top of summation sign, we keep it on bottom

LAST TIME: Proved that there is no bijection between 01REALS -- reals in [0,1) -- and N
Proof by was by contradiction using **diagonalization technique**

\[
\begin{array}{c|c}
 n & f(n) \\
 \hline
 1 & 0.3141592653 \ldots \\
 2 & 0.3737373737 \ldots \\
 3 & 0.1428571428 \ldots \\
 4 & 0.7071067811 \ldots \\
 5 & 0.3750000000 \ldots \\
 \vdots & \vdots 
\end{array}
\]

0.37210 \ldots 0.48321 \ldots

This technique is always used as part of a proof by contradiction
**proof by diagonalization** - proof by contradiction using diagonalization technique

Here is another example of **proof by diagonalization**:

**Claim**: there is no bijection between N and P(N)

**Note**: P(N) uses BOTH FINITE AND INFINITE SETS

**Proof**:

1. suppose such a bijection **exists**, call it \( m \)
2. list all elements of P(N) in the following order:
   \( m(0), m(1), m(2), m(3), m(4) \ldots \) -- these are sets of naturals
3. this will be a **complete list of P(N)**, and every element of P(N) will have a finite index
4. consider the following infinite set of naturals D':
   \[ \{ k : k \in m(k) \} \]
5. now consider the following set $D'$:

$$\{k : k \notin m(k)\}$$

6. NOTE: for any $i$, $m(i)$ differs from $D'$ with respect to element $i$: if $i$ is in $m(i)$, it is not in $D'$ and vice versa.

7. Therefore, $D'$ cannot be in this list.

8. Note that $D'$ is a member of $P(\mathbb{N})$, and we concluded in (3) that it must be in this list.

9. THERE IS A CONTRADICTION BETWEEN 7 AND 8!!! Therefore our initial assumption is false, and no such bijection is possible!!!

QED

Here is illustration of this proof, where $k$'th value is shown as 1 when $k$ is in the set, and 0 if it's not. $D$ is shown in red, and $D'$ in blue.

\[
\begin{array}{c}
s_1 = 0000000000000000... \\
s_2 = 1111111111111111... \\
s_3 = 0101010101010101... \\
s_4 = 1010101010101010... \\
s_5 = 1101011010101010... \\
s_6 = 0011011011011011... \\
s_7 = 1000100100100100... \\
s_8 = 0011000110011001... \\
s_9 = 1100110011001100... \\
s_{10} = 1101110011001100... \\
s_{11} = 1101010010010010... \\
... \\
s = 101110100111... \\
\end{array}
\]

It follows from last claim that $|\mathbb{N}| \neq |P(\mathbb{N})|$

And since there is no bijection between $\mathbb{N}$ and $01REALS$ (last time), we also know that $|\mathbb{N}| \neq |01REALS|$

In each case one of them is strictly larger than the other -- which one??

Recall, last time, we saw 1-1 mapping from $\mathbb{N}$ to $01REALS$, so $|\mathbb{N}| \leq |01REALS|$

FROM BOTH OF THESE FACTS (that $|\mathbb{N}| \neq |01REALS|$, and $|\mathbb{N}| \leq |01REALS|$) it follows that $|\mathbb{N}| < |01REALS|$
We just proved that $|N| \neq |P(N)|$ (proof by diagonalization).

But there is a 1-1 mapping $f$ from $N$ to $P(N)$: $f(k) \rightarrow \{k\}$! $\implies |N| \leq |P(N)|$

FROM BOTH OF THESE FACTS (that $|N| \neq |P(N)|$ and $|N| \leq |P(N)|$) it follows that $|N| < |P(N)|$.

Recall, last time saw a bijection between $P(N)$ and infinite bit strings (we'll call that set IBS); therefore $|IBS| = |P(N)|$.

Here is what we have so far:

$\aleph_0 = |N| < |P(N)| = |IBS| = |01REALS| = |REALS|

Note that one can find a 1-1 mapping from IBS to 01REALS.

HINT: think of those reals in $[0,1)$ which only have digits 0 and 1.

So $|IBS| \leq |01REALS|$

And one can also find 1-1 mapping from 01REALS to IBS.

HINT: THINK binary representation of digits....

So $|01REALS| \leq |IBS|$

We therefore conclude that $|IBS| = |01REALS|$

$\aleph_0 = |N| < |P(N)| = |IBS| = |01REALS|$

How to connect it to reals?

Note that one can find a 1-1 mapping from 01reals $\rightarrow$ reals, $f(x) = x$

And one can also find a 1-1 mapping from reals to 01reals.

HINT: use even digits for part after decimal point, odd digits for part before.

This is an **interleaving** technique.

We have seen interleaving before when mapping integers to $N$. 

We therefore conclude that $|\mathbb{01REALS}| = |\mathbb{REALS}|$

\[\aleph_0 = |\mathbb{N}| < |\mathbb{P(N)}| = |\mathbb{IBS}| = |\mathbb{01REALS}| = |\mathbb{01REALS}| = |\mathbb{REALS}|\]

**Infinite hierarchy of strictly larger cardinalities**

- $\mathbb{N}$ - cardinality $= \aleph_0$
- $\mathbb{P(N)}$ - cardinality $= \aleph_1$
- $\mathbb{P(P(N))}$ - cardinality $= \aleph_2$
  ...

**Cardinality math**

We have seen addition and multiplication

\[\aleph_0 + \aleph_0 = \aleph_0\]
\[\aleph_0 \times \aleph_0 = \aleph_0\]

Now we also have **exponentiation**

- $\mathbb{P(N)}$ - cardinality $= 2^{\aleph_0} = \aleph_1$
- $\mathbb{P(P(N))}$ - cardinality $= 2^{\aleph_1} = \aleph_2$
  etc etc

3 types of sets of interest to us:

- finite, countable ($\aleph_0$), uncountable ($\aleph_1$ and above)

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**SEQUENCES - HAVE BEEN SEEING THEM A LOT RECENTLY**

A collection of numbers, with order, usually written with indices

Finite sequence $S_{\text{fin}}$ of size $n+1$: $S_0, S_1, S_2, \ldots, S_n$

Infinite sequence $S_{\text{inf}}$: $S_0, S_1, S_2, S_3, \ldots$

NOTE: $S_{\text{inf}}$ has a bijection with naturals, where the index of each element in $S$ gives you the corresponding natural
Sequences are usually defined by a mapping from naturals to corresponding sequence element

**EXAMPLE:** \( f(k) = \text{sum}(0..k) \) -- defines an infinite sequence

\[
\begin{align*}
S_0 &= f(0) = \text{sum}(0..0) = 0 \\
S_1 &= f(1) = \text{sum}(0..1) = 1 \\
S_2 &= f(2) = \text{sum}(0..2) = 3 \\
&\vdots \\
S_5 &= f(5) = \text{sum}(0..5) = 5+4+3+2+1+0 = 15 \\
&\vdots
\end{align*}
\]

**FORMULA FOR** \( S_k \) ???

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & \ldots & k-2 & k-1 & k \\
k & k-1 & k-2 & k-3 & \ldots & 2 & 1 & 0
\end{array}
\]

\[ -- \text{sum is } S_k \]

\[
\begin{array}{cccccccc}
k & k & k & k & k & k & k & k \\
\end{array}
\]

\[ -- \text{sum is } 2 \ S_k = k \times (k+1) \]

Therefore, \( S_k = k \times (k+1) / 2 \)

**NOTE:** \( S_k = \text{sum}(0..k) \) can also be defined **recursively**!

- **base case:** \( S_k = 0 \) when \( k = 0 \)
- **recursive case:** \( S_k = S_{k-1} + k \) when \( k > 0 \)

Other recursive definitions of sequences:

- **Fact** \( k \) = \( \text{fact}(k) = \text{product}(1...k) \) defined recursively too
- **Fib** \( k \) = \( \text{fibonacci}(k) \)

Whenever something is defined recursive, you can do **recursive proofs** over it!! Mathematicians call it **induction**, which is really just recursion

**Claim:** \( S_k = k \times (k+1) / 2 \)

**PROOF** (recursive proof, follows the recursive definition)

1. **Base case:** (plug in the value to show the claim is true)

   // for base case, we need to prove that \( S_k = k \times (k+1) / 2 \) for the case when \( k=0 \)
so we PLUG IT IN on left and on right:

\[ S_0 = \text{sum}(0..0) = 0 \]

\[ k \times (k+1) / 2 = 0 \times (0 + 1) / 2 = 0 \]

Therefore, \[ s(k) = k \times (k+1) / 2 \] when \[ k = 0 \]

2. **Inductive assumption**: (TRUST RECURSION)
Assume that the claim is true for \( k-1 \), meaning that \( S_{k-1} = \text{sum}(0..k-1) = (k-1) \times k / 2 \)

3. **Recursive step**: (Use inductive assumption to prove the actual claim)
We show that \( S_k = k \times (k+1) / 2 \) as follows:

\[ S_k = \text{sum}(0..k) = \text{sum}(0..k-1) + k = (k-1) \times k / 2 + k = (k-1) \times k / 2 + 2k / 2 = k \times (k+1) / 2 \]

QED