Reading (in textbook, unless stated otherwise):

- Section 7.4 through example 7.4.4

NOTE: where the book differs from lecture material, please follow the lecture material instead

**DEFINITION**: (review) A set S is countable – there is a bijection between S and \( \mathbb{N} = \{0, 1, 2, 3, 4...\} \)

**DEFINITION**: (review) Bijection – a mapping that is total function, one-to-one and onto

**Claim**: \( \mathbb{N} \) is countable

**Proof** (need bijection from \( \mathbb{N} \) to \( \mathbb{N} \))
1. consider the IDENTITY MAPPING, that maps every element to itself: \( 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 2 \)
2. this is a bijection from \( \mathbb{N} \) to \( \mathbb{N} \) because it is total function, one-to-one and onto
3. so \( \mathbb{N} \) is countable

**Note**: It can be proved that all identity mappings are bijections

**Note**: It can be proved an inverse of an identity mapping is also an identity mapping

How about the set \( \mathbb{N}^+ = \{1, 2, 3, 4...\} \)? Is \( \mathbb{N}^+ \) countable?

We are looking for a bijection from \( \mathbb{N}^+ \) to \( \mathbb{N} \)

\( x \to x-1 \) is bijection from \( \mathbb{N}^+ \) to \( \mathbb{N} \), so \( \mathbb{N}^+ \) is countable

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

How about Evens = \( \{0, 2, 4, 6, \ldots\} \)? Is Evens countable?

We are looking for a bijection from Evens to \( \mathbb{N} \)

\( x \to x/2 \) is bijection from Evens to \( \mathbb{N} \), so Evens is countable

\[
\begin{array}{cccc}
0 & 2 & 4 & 6 \\
1 & 2 & 3 & 4
\end{array}
\]

How about integers = \( \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \), are integers countable?

We are looking for a bijection from integers to \( \mathbb{N} \)

\( x \to -x \) ?? NO, the codomain is not \( \mathbb{N} \)

\( x \to |x| \) ?? Hint: \( 5 \to 5, -5 \to 5 \) NOT ONE-TO-ONE!
Here is the mapping: \( x \rightarrow 2x \) for positive integers, \( x \rightarrow -2x -1 \) for negative integers

<table>
<thead>
<tr>
<th>( 0 )</th>
<th>( -1 )</th>
<th>( 1 )</th>
<th>( -2 )</th>
<th>( 2 )</th>
<th>( -3 )</th>
<th>( 3 )</th>
<th>( -4 )</th>
<th>( 4 )</th>
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</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( 1 )</td>
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<td>( 5 )</td>
<td>( 6 )</td>
<td>( 7 )</td>
<td>( 8 )</td>
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</table>

This is a bijection, therefore integers are countable.

How about pairs of naturals \( \mathbb{N} \times \mathbb{N} = \{(0, 0), (0,1), (0, 2), ... \) \( (1, 0), (1, 1), (1, 2),... (2, 0), (2, 1), (2,2)\} \)

Is \( \mathbb{N} \times \mathbb{N} \) countable? Is there a bijection \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \)?

One idea: map each pair to its sum

- total function? yes
- onto? yes
- one-to-one? NO!! BAD IDEA :( 

HERE IS THE IDEA for this mapping:

A. List all the pairs in the following order:
   0. pairs whose sum is 0
   1. pairs whose sum is 1
   2. pairs whose sum is 2

B. Within each group, list them by increasing first member of the pair.

C. We map each pair to its index into this list

Here is how it looks:

\[(0,0) (0,1) (1, 0) (0,2) (1, 1) (2, 0) (0,3) (1,2) (2,1) (3,0) (0,4) (1,3) (2,2) (3,1) \].....

Note: each group of pairs is finite

\[(0, k), (1, k-1), (2, k-2) ... (k, 0)\]

In fact, for any \( k \), if you consider all pairs whose sum is \( k \), there are \( k+1 \) such pairs

0. pairs whose sum is 0 \( \rightarrow \) 1 pair
1. pairs whose sum is 1 \( \rightarrow \) 2 pairs
2. pairs whose sum is 2 \( \rightarrow \) 3 pairs

This mapping is a total function, one to one and onto!

So this mapping \( \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) is a bijection, therefore \( \mathbb{N} \times \mathbb{N} \) is also countable!!

How do you actually compute the natural that corresponds to a given pair?

1. First, consider the index of the FIRST pair in every group

\[(0,0) (0,1) (1, 0) (0,2) (1, 1) (2, 0) (0,3) (1,2) (2,1) (3,0) (0,4) (1,3) (2,2) (3,1) \].....

\[0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13\]....

\[\wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \quad \wedge \]
2. Note that for any k, the index for FIRST pair in k'th group is the sum from 0 to k
   
   \[0..0 \rightarrow 0, \quad 0..1 \rightarrow 1, \quad 0..2 \rightarrow 3, \quad 0..3 \rightarrow 1+2+3 = 6, \quad 0..4 = 1+2+3+4 = 10, \ldots\]
   
   The formula for the sum from 0 to k is \( k(k+1)/2 \) (we will prove this later in the course)

3. Then use the 1st member of the pair to index into that group, adding it to the other index

So in general if we have (a, b) as the pair, its sum is a+b, which means that it's in group a+b

Then its corresponding natural = \((a+b)(a+b+1)/2 + a\)

Last example of countable set: \(S = \) finite strings over \{a..z\} (26 chars)

\(S\) includes "", "dina", "a", "abcdeabcdeabcdeabcdeabcdeabcdeabcde"....

What is the a bijection from \(S\) to \(N\)?

Alternately, can we list all member of \(S\) so each one corresponds uniquely so some index [0,1,2,3...]

Note that the indices in that list give you the corresponding naturals!

Here is a guess:

1. list all strings that start with a, alphabetically
   
   "a", "aa", "aaa", "aaaa", ......

2. list all strings that start with b, alphabetically
3. list all strings that start with c, alphabetically

Problem: There are INFINITELY MANY strings in the first group, including "a", "aa", "aaa", "aaaa", ......

So what is the index of "b"? NOT A NATURAL

BUT WE CAN SORT BY LENGTH!

So here is a better approach:

0. list all strings of length 0, ""

1. list all strings of length 1, alphabetically
2. list all strings of length 2, alphabetically

Note that each group is finite

And this is a bijection \(S \rightarrow N\), where the indices in that list give you the corresponding naturals!

Therefore, \(S\) is countable

**Note:** NOT ALL SETS ARE COUNTABLE

**Example:** reals, infinite strings (we'll prove this next time)
MEASURING INFINITY

For finite sets, their size is a natural number
Infinite sets, their size is infinity $\infty$
But there are different infinities! Meaning, not all infinite sets are the same size

SET CARDINALITY - like set size, but includes infinities
$|S|$ = cardinality of S

Smallest infinity is the size of countable sets = $\aleph_0$ "aleph-null"
$\aleph_0$ not a number, but acts in many ways like a number

The set of reals has size $\aleph_1$, and we will show that $\aleph_0 < \aleph_1$

Important rules:
• If there is a bijection between any sets A and B, then they have the same cardinality, $|A| = |B|
• If $A \subseteq B$, then we say that $|A| \leq |B|

Note that $A \subset B$ does not work as you might expect
If $A \subset B$ and A is finite, then $|A| < |B|
But for infinite A, it will not necessarily be the case that $|A| < |B|$; all you can say is that $|A| \leq |B|

Example: $A = \text{Evens}$ $B = \text{Naturals}$
$A \subset B$ because Evens is a strict subset of Naturals
But we saw a bijection between them, proving that A is countable $|A| = |B| = \aleph_0$
So it cannot be the case that $|A| < |B|$ here

If there is a bijection between any sets A and B, there is a bijection from B to A, $|B| = |A|

$|A| = |B|$ iff $|B| = |A|
$|A| \leq |B|$ iff $|B| \geq |A|
$|A| < |B|$ iff $|B| > |A|$
Given sets A, B, C, we can establish relationship between |A| and |C| using |B|

**Example 1:** there is a bijection between A and B, and bijection from B to C

|A| = |B|, |B| = |C|

can conclude that |A| = |C|

**Example 2:** there is a bijection between A and B; B is a subset of C

|A| = |B|, |B| $\leq$ |C|

can conclude that |A| $\leq$ |C|

**Example 3:** B is a subset of A and there is a bijection between B and C

|A| $\geq$ |B|, |B| = |C|

can conclude that |A| $\geq$ |C|

**INFINITE HOTEL PARADOX**

We have a hotel with infinitely many rooms numbered 0, 1, 2, 3, 4, ....

We get infinitely many guests that fill up all the rooms...

Then 1 more guest shows up!! how can we find them an EMPTY room?

Solution: move everyone up a room $x \rightarrow x+1$ and put new guest in room 0!

Then 1000 more guests shows up!! how can we find them all an EMPTY room?

Solution: move everyone: $x \rightarrow x+1000$

Now, infinitely many new guests show up!! (countably infinite $\mathbb{N}_0$)

Solution: move everyone: $x \rightarrow 2 \times x$